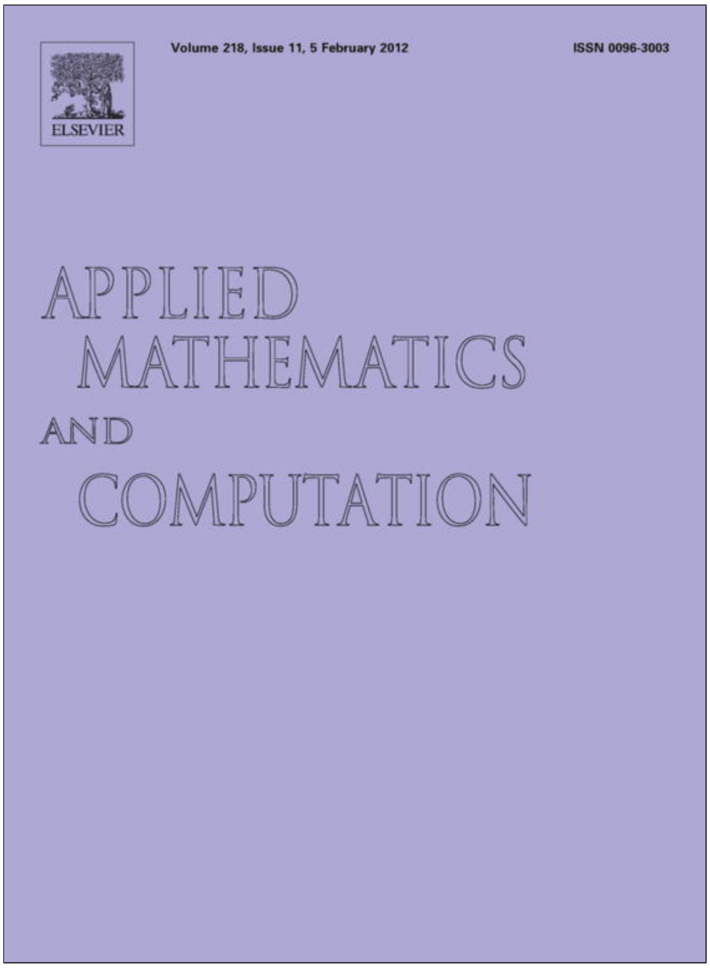
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Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane ☆

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ABSTRACT

A normalized analytic function f defined on the open unit disk in the complex plane is in the class $\mathcal{SL}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2-1|<1$. In the present investigation, the $\mathcal{SL}$-radii for certain well-known classes of functions are obtained. Radius problems associated with the left-half plane are also investigated for these classes.

1. Introduction

Let $\mathcal{A}_n$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ of the form $f(z) = z + \sum_{k=n+1} a_k z^k$, and let $\mathcal{A} := \mathcal{A}_1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{SL}$ be the class of functions defined by

$$\mathcal{SL} := \left\{ f \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\} \quad (z \in \mathbb{D}).$$

Thus a function $f \in \mathcal{SL}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. For two functions f and g analytic in $\mathbb{D}$, the function f is said to be *subordinate* to g, written $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. In terms of subordination, the class $\mathcal{SL}$ consists of normalized analytic functions f satisfying $zf'(z)/f(z) \prec \sqrt{1+z}$. This class $\mathcal{SL}$ was introduced by Sokół and Stankiewicz [20]. Paprocki and Sokół [10] discussed a more general class $\mathcal{S}^*(a,b)$ consisting of normalized analytic functions f satisfying $|[zf'(z)/f(z)]^a - b| < b$, $b \geqslant 1/2$, $a \geqslant 1$.

Recall that a function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is starlike with respect to 0. Similarly, a function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex. Analytically, a function $f \in \mathcal{A}$ is starlike or convex if the following respective subordinations hold:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}.$$

Ma and Minda [6] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $(1+z)/(1-z)$ by a more general function φ. They considered analytic univalent functions φ with positive real part that map the unit disk $\mathbb{D}$ onto regions starlike with respect to 1, symmetric with respect to the real axis and normalized by $\varphi(0) = 1$. They introduced the following classes that include several well-known classes as special cases:

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$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad \text{and} \quad \mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

For $0 \leqslant \alpha < 1$,

$$\mathcal{ST}(\alpha) := \mathcal{ST}((1 + (1 - 2\alpha)z)/(1 - z)), \quad \mathcal{CV}(\alpha) := \mathcal{CV}((1 + (1 - 2\alpha)z)/(1 - z))$$

are the subclasses of $\mathcal{S}$ consisting of starlike and convex functions of order α in $\mathbb{D}$, respectively. Then $\mathcal{ST} := \mathcal{ST}(0)$, $\mathcal{CV} := \mathcal{CV}(0)$ are the well-known classes of starlike and convex functions, respectively. Also let

$$\mathcal{ST}_n(\alpha) := \mathcal{A}_n \cap \mathcal{ST}(\alpha), \quad \mathcal{CV}_n(\alpha) := \mathcal{A}_n \cap \mathcal{CV}(\alpha), \quad \mathcal{SL}_n := \mathcal{A}_n \cap \mathcal{SL}.$$

Since $\mathcal{SL} = \mathcal{ST}(\sqrt{1+z})$, distortion, growth, and rotation results for the class $\mathcal{SL}$ can conveniently be obtained by applying the corresponding results in [6].

The radius of a property P in a set of functions $\mathcal{M}$, denoted by $R_P(\mathcal{M})$, is the largest number R such that every function in the set $\mathcal{M}$ has the property P in each disk $\mathbb{D}_r = \{z \in \mathbb{D} : |z| < r\}$ for every $r < R$. For example, the radius of convexity in the class $\mathcal{S}$ is $2 - \sqrt{3}$. Sokół and Stankiewicz [20] determined the radius of convexity for functions in the class $\mathcal{SL}$. They also obtained structural formula, growth and distortion theorems for these functions. Estimates for the first few coefficients of functions in this class can be found in [21]. Recently, Sokół [22] determined various radii for functions belonging to the class $\mathcal{SL}$; these include the radii of convexity, starlikeness and strong starlikeness of order α. In contrast, in our present investigation, we compute the $\mathcal{SL}$-radius for functions belonging to several interesting classes. Unlike the radii problems associated with starlikeness and convexity, where a central feature is the estimates for the real part of the expressions $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$, respectively, the $\mathcal{SL}$-radius problems for classes of functions are tackled by first finding the disk that contains the values of $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$. This approach was earlier used for the class of uniformly convex functions investigated in [3–5,12–19]. The technical result required will be presented in the next section.

Another interesting class is $\mathcal{M}(\beta), \beta > 1$, defined by

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) < \beta, \ z \in \Delta \right\}.$$

The class $\mathcal{M}(\beta)$ was investigated by Uralegaddi et al. [23], while its subclass was investigated by Owa and Srivastava [9]. We let $\mathcal{M}_n(\beta) := \mathcal{A}_n \cap \mathcal{M}(\beta)$. In the present paper, radius problems related to $\mathcal{M}(\beta)$ will also be investigated. Related radius problem for this class can be found in [1,2,11]. The following definitions and results will be required.

An analytic function $p(z) = 1 + c_n z^n + \cdots$ is a function with positive real part if $\operatorname{Re} p(z) > 0$. The class of all such functions is denoted by $\mathcal{P}_n$. We also denote the subclass of $\mathcal{P}_n$ satisfying $\operatorname{Re}\ p(z) > \alpha$, $0 \leqslant \alpha < 1$, by $\mathcal{P}_n(\alpha)$. More generally, for $-1 \leqslant B < A \leqslant 1$, the class $\mathcal{P}_n[A, B]$ consists of functions p of the form $p(z) = 1 + c_n z^n + \cdots$ satisfying

$$p(z) \prec \frac{1 + Az}{1 + Bz}.$$

Lemma 1.1 [7]. *If $p \in \mathcal{P}_n$, then*

$$\left| \frac{zp'(z)}{p(z)} \right| \leqslant \frac{2nr^n}{1 - r^{2n}} \quad (|z| = r < 1).$$

Lemma 1.2 [12]. *If $p \in P_n[A, B]$, then*

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leqslant \frac{(A - B)r^n}{1 - B^2 r^{2n}} \quad (|z| = r < 1).$$

In particular, if $p \in P_n(\alpha)$, then

$$\left| p(z) - \frac{1 + (1 - 2\alpha)r^{2n}}{1 - r^{2n}} \right| \leqslant \frac{2(1 - \alpha)r^n}{1 - r^{2n}} \quad (|z| = r < 1).$$

2. The $\mathcal{SL}_n$-radius problems

In this section, three special classes of functions will be considered. First motivated by MacGregor [7,8], is the class

$$\mathcal{S}_n := \left\{ f \in \mathcal{A}_n : \frac{f(z)}{z} \in \mathcal{P}_n \right\}.$$

For this class, we shall find its $\mathcal{SL}_n$-radius, denoted by $R_{\mathcal{SL}_n}(\mathcal{S}_n)$.

Theorem 2.1. *The $\mathcal{SL}_n$-radius for the class $\mathcal{S}_n$ is*

$$R_{\mathcal{SL}_n}(\mathcal{S}_n) = \left\{ \frac{\sqrt{2}-1}{n+\sqrt{n^2+(\sqrt{2}-1)^2}} \right\}^{1/n}.$$

This radius is sharp.

Proof. Let $f \in \mathcal{S}_n$. Define the function h by

$$h(z) = \frac{f(z)}{z}.$$

Then the function $h \in \mathcal{P}_n$ and

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zh'(z)}{h(z)}.$$

Applying Lemma 1.1 to the function h yields

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leqslant \frac{2nr^n}{1-r^{2n}}.$$

Notice that if $|w-1| \leqslant \sqrt{2}-1$, then $|w+1| \leqslant \sqrt{2}+1$ and hence $|w^2-1| \leqslant 1$. Thus the disk $|w-1| \leqslant 2nr^n/(1-r^{2n})$ lies inside the lemniscate $|w^2-1| \leqslant 1$ if

$$\frac{2nr^n}{1-r^{2n}} \leqslant \sqrt{2}-1.$$

Solving this inequality for r yields

$$r \leqslant R := \left\{ \frac{\sqrt{2}-1}{n+\sqrt{n^2+(\sqrt{2}-1)^2}} \right\}^{1/n}.$$

To show that the above upper bound cannot be increased and so R is the $\mathcal{SL}_n$-radius for the class $\mathcal{S}_n$, consider the function f defined by

$$f(z) = \frac{z+z^{n+1}}{1-z^n}.$$

Clearly the function f satisfies the hypothesis of the theorem and

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2nz^n}{1-z^{2n}}.$$

At $z = R$, routine computations show that

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| = \left| \left(1 + \frac{2nR^n}{1-R^{2n}} \right)^2 - 1 \right| = 1.$$

This proves that R is the $\mathcal{SL}_n$-radius for the class $\mathcal{S}_n$ and that the result is sharp. □

The following technical lemma will be useful in our subsequent investigations.

Lemma 2.2. *For $0 < a < \sqrt{2}$, let r_a be given by*

$$r_a = \begin{cases} \left(\sqrt{1-a^2} - (1-a^2) \right)^{1/2} & (0 < a \leqslant 2\sqrt{2}/3), \\ \sqrt{2} - a & (2\sqrt{2}/3 \leqslant a < \sqrt{2}) \end{cases}$$

and for $a > 0$, let R_a be given by

$$R_a = \begin{cases} \sqrt{2} - a & (0 < a \leqslant 1/\sqrt{2}), \\ a & (1/\sqrt{2} \leqslant a). \end{cases}$$

Then

$$\{w : |w - a| < r_a\} \subseteq \{w : |w^2 - 1| < 1\} \subseteq \{w : |w - a| < R_a\}.$$

Proof. The equation of the lemniscate of Bernoulli is

$$(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$$

and the parametric equations of its right-half is given by

$$x(t) = \frac{\sqrt{2}\cos t}{1 + \sin^2 t}, \quad y(t) = \frac{\sqrt{2}\sin t \cos t}{1 + \sin^2 t}, \quad \left(-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}\right).$$

The square of the distance from the point $(a,0)$ to the points on the lemniscate is given by

$$z(t) = (a - x(t))^2 + (y(t))^2 = a^2 + \frac{2(\cos^2 t - \sqrt{2}a\cos t)}{1 + \sin^2 t}$$

and its derivative is

$$z'(t) = 2\frac{(-4\cos t + \sqrt{2}a(2 + \cos^2 t))\sin t}{(1 + \sin^2 t)^2}.$$

Clearly $z'(t) = 0$ if and only if

$$t = 0 \quad \text{or} \quad \cos t = \frac{\sqrt{2}(1 \pm \sqrt{1 - a^2})}{a}.$$

Note that for $a > 1$, the numbers $\sqrt{2}(1 \pm \sqrt{1 - a^2})/a$ are complex and for $0 < a \leqslant 1$, the number $\sqrt{2}(1 + \sqrt{1 - a^2})/a > 1$. For $0 < a < 1$, the number $\sqrt{2}(1 - \sqrt{1 - a^2})/a$ lies between -1 and 1 if and only if $0 < a \leqslant 2\sqrt{2}/3$.

Let us first assume that $0 < a \leqslant 2\sqrt{2}/3$ and $t = t_0$ be given by

$$\cos t_0 = \frac{\sqrt{2}(1 - \sqrt{1 - a^2})}{a}.$$

Since

$$\min\{z(\pi/2), z(-\pi/2), z(0), z(t_0)\} = z(t_0),$$

it follows that $\min\sqrt{z(t)} = \sqrt{z(t_0)}$. A calculation shows that

$$z(t_0) = \sqrt{1 - a^2} - (1 - a^2).$$

Hence

$$r_a = \min\sqrt{z(t)} = \sqrt{\sqrt{1 - a^2} - (1 - a^2)}.$$

Let us next assume that $2\sqrt{2}/3 \leqslant a < \sqrt{2}$. In this case,

$$\min\{z(\pi/2), z(-\pi/2), z(0)\} = z(0)$$

and thus $z(t)$ attains its minimum value at $t = 0$ and

$$r_a = \min\sqrt{z(t)} = \sqrt{2} - a.$$

Now consider $0 < a \leqslant 1/\sqrt{2}$ and $t = t_0$ be given by

$$\cos t_0 = \frac{\sqrt{2}(1 - \sqrt{1 - a^2})}{a}.$$

It is easy to see that

$$\max\{z(\pi/2), z(-\pi/2), z(0), z(t_0)\} = z(0)$$

and thus

$$R_a = \max\sqrt{z(t)} = \sqrt{2} - a.$$

Similarly, for $a \geqslant 1/\sqrt{2}$,

$$\max\{z(\pi/2), z(-\pi/2), z(0)\} = z(\pi/2)$$

and hence

$$R_a = \max \sqrt{z(t)} = a. \qquad \square$$

Now consider the subclass $\mathcal{CS}_n(\alpha)$ consisting of close-to-starlike functions of type α defined by

$$\mathcal{CS}_n(\alpha) := \left\{ f \in \mathcal{A}_n : \frac{f}{g} \in \mathcal{P}_n,\ g \in \mathcal{ST}_n(\alpha) \right\}.$$

The $\mathcal{SL}_n$-radius for this class is given in the following theorem.

Theorem 2.3. *The $\mathcal{SL}_n$-radius for the class $\mathcal{CS}_n(\alpha)$ is given by*

$$R_{\mathcal{SL}_n}(\mathcal{CS}_n(\alpha)) = \left(\frac{\sqrt{2}-1}{(1+n-\alpha) + \sqrt{(1+n-\alpha)^2 + (1-2\alpha+\sqrt{2})(\sqrt{2}-1)}} \right)^{1/n}$$

This radius is sharp.

Proof. Let g be a starlike function of order α with $h(z) = f(z)/g(z) \in \mathcal{P}_n$. Then $zg'(z)/g(z)$ is in $\mathcal{P}_n(\alpha)$ and from Lemma 1.2,

$$\left| \frac{zg'(z)}{g(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}} \right| \leqslant \frac{2(1-\alpha)r^n}{1-r^{2n}}. \tag{2.1}$$

Applying Lemma 1.1 yields

$$\left| \frac{zh'(z)}{h(z)} \right| \leqslant \frac{2nr^n}{1-r^{2n}}. \tag{2.2}$$

Now

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} \tag{2.3}$$

and using (2.1)–(2.3), it follows that

$$\left| \frac{zf'(z)}{f(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}} \right| \leqslant \frac{2(1+n-\alpha)r^n}{1-r^{2n}}. \tag{2.4}$$

Since the center of the disk in (2.4) is greater than 1, from Lemma 2.2, it is seen that the points w are inside the lemniscate $|w^2-1| \leqslant 1$ if

$$\frac{2(1+n-\alpha)r^n}{1-r^{2n}} \leqslant \sqrt{2} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}.$$

The last inequality reduces to $(1-2\alpha+\sqrt{2})r^{2n} + 2(1+n-\alpha)r^n - (\sqrt{2}-1) \leqslant 0$. Solving this latter inequality results in the value of $R = R_{\mathcal{SL}_n}(\mathcal{CS}_n(\alpha))$.

The function f given by

$$f(z) = \frac{z(1+z^n)}{(1-z^n)^{(n+2-2\alpha)/n}}$$

satisfies the hypothesis of Theorem 2.3 with $g(z) = z/(1-z^n)^{(2-2\alpha)/n}$. It is easy to see that, for $z = R = R_{\mathcal{SL}_n}(\mathcal{CS}_n(\alpha))$,

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| = \left| \frac{[1+(1-2\alpha)R^{2n} + 2(1+n-\alpha)R^n]^2}{(1-R^{2n})^2} - 1 \right| = 1.$$

This shows that the result is sharp. $\square$

For $-1 \leqslant B < A \leqslant 1$, define the class

$$\mathcal{ST}_n[A,B] := \left\{ f \in \mathcal{A}_n : \frac{zf'(z)}{f(z)} \in \mathcal{P}_n[A,B] \right\}.$$

The class $\mathcal{ST}_1[A,B]$ is the well-known class of Janowski starlike functions. For the class $\mathcal{ST}_n[A,B]$, the $\mathcal{SL}_n$ radius is investigated in Theorems 2.4, 2.5, and 2.7; Theorem 2.4 investigates the conditions on A and B for the $\mathcal{SL}_n$ radius to be 1 while Theorems 2.5 and 2.7, respectively deal with the cases $B \leqslant 0$ and $B > 0$.

Theorem 2.4. *Let* $-1 < B < A \leqslant 1$ *and either (i)* $1 + A \leqslant \sqrt{2}(1 + B)$ *and* $2\sqrt{2}(1 - B^2) \leqslant 3(1 - AB) < 3\sqrt{2}(1 - B^2)$, *or (ii)* $(A - B)(1 - B^2) + (1 - B^2)^2 \leqslant (1 - B^2)\sqrt{(1 - B^2) - (1 - AB)^2} + (1 - AB)^2$ *and* $2\sqrt{2}(1 - B^2) \geqslant 3(1 - AB)$. *Then* $\mathcal{ST}_n[A,B] \subset \mathcal{SL}_n$.

Proof. Since $\frac{zf'(z)}{f(z)} \in P_n[A,B]$, Lemma 1.2 gives

$$\left|\frac{zf'(z)}{f(z)} - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2} \quad (|z| < 1). \tag{2.5}$$

Let $a = (1 - AB)/(1 - B^2)$, and suppose the two conditions in (i) hold. By multiplying the inequality $1 + A \leqslant \sqrt{2}(1 + B)$ by the positive constant $1 - B$ and rewriting, it is seen that the given inequality is equivalent to $A - B \leqslant \sqrt{2}(1 - B^2) - (1 - AB)$. A division by $1 - B^2$ shows that the condition $1 + A \leqslant \sqrt{2}(1 + B)$ is equivalent to the condition $(A - B)/(1 - B^2) \leqslant \sqrt{2} - a$. Similarly, the condition $2\sqrt{2}(1 - B^2) \leqslant 3(1 - AB) < 3\sqrt{2}(1 - B^2)$ is equivalent to $2\sqrt{2}/3 \leqslant a < \sqrt{2}$. In view of these equivalences, it follows from (2.5) that the quantity $w = zf'(z)/f(z)$ lies in the disk $|w - a| < r_a$ where $r_a = \sqrt{2} - a$. Since $2\sqrt{2}/3 \leqslant a < \sqrt{2}$ and $|w - a| < r_a$, Lemma 2.2 shows that $|w^2 - 1| < 1$ or

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1.$$

This proves that $f \in \mathcal{SL}_n$. The proof is similar if the conditions in (ii) hold, and is therefore omitted. □

Theorem 2.5. *Let* $-1 \leqslant B < A \leqslant 1$, *with* $B \leqslant 0$. *Then the* $\mathcal{SL}_n$*-radius for the class* $\mathcal{ST}_n[A,B]$ *is*

$$R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B]) = \min\left(1, \left(\frac{2(\sqrt{2} - 1)}{(A - B) + \sqrt{(A - B)^2 + 4(\sqrt{2}B - A)B(\sqrt{2} - 1)}}\right)^{\frac{1}{n}}\right).$$

In particular, if $1 + A < \sqrt{2}(1 + B)$, *then* $\mathcal{ST}_n[A,B] \subseteq \mathcal{SL}_n$. *Also the* $\mathcal{SL}$*-radius for the class consisting of starlike functions is* $3 - 2\sqrt{2}$.

Proof. Since $\frac{zf'(z)}{f(z)} \in P_n[A,B]$, Lemma 1.2 yields

$$\left|\frac{zf'(z)}{f(z)} - \frac{1 - ABr^{2n}}{1 - B^2r^{2n}}\right| \leqslant \frac{(A - B)r^n}{1 - B^2r^{2n}}.$$

Since $B \leqslant 0$, it follows that

$$a := \frac{1 - ABr^{2n}}{1 - B^2r^{2n}} \geqslant 1.$$

Using Lemma 2.2, the function f satisfies

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| \leqslant 1$$

provided

$$\frac{(A - B)r^n}{1 - B^2r^{2n}} \leqslant \sqrt{2} - \frac{1 - ABr^{2n}}{1 - B^2r^{2n}},$$

that is,

$$(\sqrt{2}B - A)Br^{2n} + (A - B)r^n - (\sqrt{2} - 1) \leqslant 0.$$

Solving the inequality, we get $r \leqslant R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B])$. The result is sharp for the function given by $f(z) = z(1 + Bz^n)^{\frac{A-B}{nB}}$ for $B \neq 0$ and $f(z) = z\exp(Az^n/n)$ for $B = 0$. Such function f satisfies the equation $zf'(z)/f(z) = (1 + Az^n)/(1 + Bz^n)$, and therefore the function $f \in \mathcal{ST}_n[A,B]$. □

Remark 2.6. Let $B < 0$. Then $1 - B^2 \leqslant 1 - AB$ and therefore $2\sqrt{2}(1 - B^2) \leqslant 3(1 - AB)$. Also the inequality $1 + A < \sqrt{2}(1 + B)$ yields $\sqrt{2}(1 - B^2) > (1 - A)(1 - B) = 1 + A - B - AB > 1 - AB$. Thus $2\sqrt{2}(1 - B^2) \leqslant 3(1 - AB) < 3\sqrt{2}(1 - B^2)$. In the case $B < 0$, Theorem 2.5 shows that the inequality $1 + A < \sqrt{2}(1 + B)$ is sufficient to deduce the inclusion $\mathcal{ST}_n[A,B] \subseteq \mathcal{SL}_n$.

Theorem 2.7. *Assume that $f \in \mathcal{ST}_n[A,B]$ and $0 < B < A \leqslant 1$. Let R_1 be given by*

$$R_1 = \left(\frac{3-2\sqrt{2}}{(3A-2\sqrt{2}B)B}\right)^{1/(2n)}$$

and let R_2 be the number $R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B])$ as given in Theorem 2.5. Let R_3 be the largest number in $(0,1]$ such that

$$(A-B)r^n(1-B^2r^{2n}) + (1-B^2r^{2n})^2 - (1-ABr^{2n})^2 - \sqrt{(1-B^2r^{2n})^2-(1-ABr^{2n})^2} \leqslant 0$$

for all $0 \leqslant r \leqslant R_3$. Then the $\mathcal{SL}_n$-radius for the class $\mathcal{ST}_n[A,B]$ is given by

$$R_{\mathcal{SL}_n}(\mathcal{ST}_n[A,B]) = \begin{cases} R_2 & (R_2 \leqslant R_1), \\ R_3 & (R_2 > R_1). \end{cases}$$

Proof. From the proof of the previous theorem, it follows that the quantity $w = zf'(z)/f(z)$ lies in the disk $|w-a| \leqslant R$, where

$$a := \frac{1-ABr^{2n}}{1-B^2r^{2n}}, \quad R = \frac{(A-B)r^n}{1-B^2r^{2n}}.$$

The $\mathcal{SL}_n$-radius is computed by finding the largest radius such that the boundary of the disk $|w-a| < R$ touches the lemniscate $|w^2-1| = 1$. When r increases from 0 to 1, the center of the disk moves from $a = 1$ to $a = (1-AB)/(1-B^2) < 1$. Depending on R, the largest disk may touch the lemniscate at $(\sqrt{2},0)$ or at two symmetrically placed points. The conditions for these two cases are given in Lemma 2.2. Note that the numbers R_1, R_2 and R_3 are determined so that $r \leqslant R_1$ if and only if $a \geqslant 2\sqrt{2}/3$, $r \leqslant R_2$ if and only if $R \leqslant \sqrt{2}-a$, and $r \leqslant R_3$ if and only if $R \leqslant (\sqrt{1-a^2}-(1-a^2))^{1/2}$.

First consider the case $R_2 \leqslant R_1$. Since $r \leqslant R_1$ is equivalent to $a \geqslant 2\sqrt{2}/3$, for $0 \leqslant r \leqslant R_2$, it follows that $a \geqslant 2\sqrt{2}/3$. From Lemma 2.2, the $\mathcal{SL}_n$ -radius satisfies the inequality $R \leqslant \sqrt{2}-a$. This shows that $f \in \mathcal{SL}_n$ in $|z| \leqslant R_2$.

Assume now that $R_2 > R_1$. In this case, since $r \geqslant R_1$ if and only if $a \leqslant 2\sqrt{2}/3$, for $r = R_2$, then $a \leqslant 2\sqrt{2}/3$. Lemma 2.2 shows that $f \in \mathcal{SL}_n$ in $|z| \leqslant r$ if $R \leqslant (\sqrt{1-a^2}-(1-a^2))^{1/2}$, or equivalently if $r \leqslant R_3$.

To prove sharpness, consider the function given by $f_0(z) = z(1+Bz^n)^{\frac{A-B}{nB}}$ if $B \neq 0$, and $f_0(z) = z \exp(Az^n/n)$ if $B = 0$. Then $\{zf'(z)/f(z)\colon |z| < r\} = \{w\colon |w-a| < R\}$, where a and R are given above, which establishes sharpness of the result. □

3. The $\mathcal{M}_n(\beta)$-radius problems

In this section, we compute the $\mathcal{M}_n(\beta)$-radii for the classes $\mathcal{S}_n$ and $\mathcal{CS}_n(\alpha)$.

Theorem 3.1. *The $\mathcal{M}_n(\beta)$-radius of functions in $\mathcal{S}_n$ is given by*

$$R_{\mathcal{M}_n(\beta)}(\mathcal{S}_n) = \left[\frac{\beta-1}{n+\sqrt{n^2+(\beta-1)^2}}\right]^{1/n}.$$

Proof. Since $h(z) = f(z)/z \in \mathcal{P}_n$, Lemma 1.1 yields

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zh'(z)}{h(z)}\right| \leqslant \frac{2nr^n}{1-r^{2n}}.$$

Therefore

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \leqslant \frac{1+2nr^n-r^{2n}}{1-r^{2n}} \leqslant \beta$$

for $r \leqslant R_{\mathcal{M}_n(\beta)}(\mathcal{S}_n)$.

The result is sharp for the function

$$f(z) = \frac{z(1+z^n)}{1-z^n},$$

which satisfies the hypothesis of Theorem 3.1. □

For the class $\mathcal{CS}_n(\alpha)$, the following radius is obtained.

Theorem 3.2. *The $\mathcal{M}_n(\beta)$-radius of functions in $\mathcal{CS}_n(\alpha)$ is given by*

$$R_{\mathcal{M}_n(\beta)}(\mathcal{CS}_n(\alpha)) = \frac{\beta - 1}{(1+n-\alpha) + \sqrt{(1+n-\alpha)^2 + (\beta-1)(1+\beta-2\alpha)}}.$$

Proof. Define the function h by

$$h(z) := \frac{f(z)}{g(z)}.$$

Then $h \in \mathcal{P}_n$ and by Lemma 1.1,

$$\left|\frac{zh'(z)}{h(z)}\right| \leqslant \frac{2nr^n}{1-r^{2n}}. \tag{3.1}$$

Since $g \in \mathcal{ST}_n(\alpha)$, it follows that $zg'(z)/g(z)$ is in $\mathcal{P}_n(\alpha)$ and therefore, by Lemma 1.2,

$$\left|\frac{zg'(z)}{g(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right| \leqslant \frac{2(1-\alpha)r^n}{1-r^{2n}}. \tag{3.2}$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)},$$

in view of (3.1) and (3.2), it is seen that

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+(1-2\alpha)r^{2n}}{1-r^{2n}}\right| \leqslant \frac{2(1+n-\alpha)r^n}{1-r^{2n}}.$$

This represents a circular disk intersecting the real axis at

$$x_0 = \frac{1-2(1+n-\alpha)r^n + (1-2\alpha)r^{2n}}{1-r^{2n}} \quad \text{and} \quad x_1 = \frac{1+2(1+n-\alpha)r^n + (1-2\alpha)r^{2n}}{1-r^{2n}}$$

and therefore

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \leqslant \frac{1+2(1+n-\alpha)r^n + (1-2\alpha)r^{2n}}{1-r^{2n}} \leqslant \beta$$

for $r \leqslant R$.

The function

$$f(z) = \frac{z(1+z^n)}{(1-z^n)^{(n+2-2\alpha)/n}}$$

satisfies the hypothesis of Theorem 3.2 with

$$g(z) = \frac{z}{(1-z^n)^{(2-2\alpha)/n}}.$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{1+2(1+n-\alpha)z^n + (1-2\alpha)z^{2n}}{1-z^{2n}} = \beta$$

for $z = R = R_{\mathcal{M}_n(\beta)}(\mathcal{CS}_n(\alpha))$, the result is sharp.

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